

Black hole solutions in $F(R)$ gravity with conformal anomaly

S. H. Hendi^{1,*} and D. Momeni^{2,†}

¹*Physics Department, College of Sciences, Yasouj University, Yasouj 75914, Iran.*

²*Department of Physics, Faculty of sciences, Tarbiat Moa'llem university, Tehran, Iran*

Abstract

In this paper, we consider $F(R) = R + f(R)$ theory instead of Einstein gravity with conformal anomaly and look for its analytical solutions. Depending on the free parameters, one may obtain both uncharged and charged solutions for some classes of $F(R)$ models. Calculation of Kretschmann scalar shows that there is a singularity located at $r = 0$, which the geometry of uncharged (charged) solution is corresponding to the Schwarzschild (Reissner-Nordström) singularity. Further, we discuss the viability of our models in details. We show that these models can be stable depending on their parameters and in different epoches of the universe.

I. INTRODUCTION

In 1970, Buchdahl considered the first modification of the Einstein Lagrangian density involving an arbitrary function of Ricci scalar, $F(R)$ [1]. After that and motivated by inflationary scenarios, Starobinsky proposed an action which is to weigh up the effects of R^2 corrections to Einstein gravity [2]. In addition, the recent advent of new observational precision tests, such as rotation curves of spiral galaxies [3] and solar system tests [4] have changed the modern view of cosmos based on General Relativity. Amongst the modification of Einstein gravity, the so-called $F(R)$ gravity is completely special (see [5] and references therein and for a review see [6]). Interpretation of the hierarchy and singularity problems [7–9], early-time inflation [10], gravitational waves detection [11] and also the four cosmological phases [12], have been investigated in $F(R)$ models.

On the other side, the conformal anomaly (CA) [13, 14] plays a crucial role in string theory [15] and gravitational field theory [16]. Also, it has been shown that there is a deep relation between Hawking radiation and trace anomaly [17]. The relation of CA with some cosmological problems such as inflation and cosmological constant problem have been considered in Refs. [14, 18]. Applications of CA in black hole and particle physics have been studied in [19–21] and [22], respectively (for a suitable treatment of CA in the black hole solutions see [20]). In addition, we should note that the first Schwarzschild-dS solutions in $F(R)$ gravity has been discussed in [23].

In the semiclassical framework of general relativity, one may use quantum field theory (QFT) to describe the matter while the gravitational field is considered as a classical field. When classical gravity is regarded as a background for a conformally coupled QFT, the trace of the classical expectation value of matter field energy momentum tensor does not vanish. In other word, considering the quantum expectation value of the energy momentum tensor during the process of renormalization, leads to additional terms in the Einstein-Hilbert action.

It has been shown that, when $F''(R) \neq 0$, metric $F(R)$ gravity is completely equivalent to a special case of scalar-tensor theory [24]. In order to find a scalar-tensor representation of $F(R)$ theories, one can use conformal transformations [24]. Considering suitable conformal transformations, one may find that, $F(R) = R + f(R)$ behaves geometrically as if we have $F(R) = R +$ (non-minimal scalar field). Remarkably, $f(R)$ plays the role of "scalar field" so that the geometry $F(R) = R + f(R)$ becomes equivalent to the Einstein-dilaton geometry in a spherically symmetry spacetime. It is obvious that the CA may come from the quantum effects of matter field. Regarding the scalar-tensor representation of $F(R) = R + f(R)$ theories, one may consider $f(R)$ (or equivalently "scalar field") as a matter field.

It is notable that there are in fact two distinct types of trace anomalies with different geometric and physical antecedents. The first anomaly called type A anomaly which is related to the Euler characteristic of $2n$ -dimensional spacetime and the second named B anomaly with a connection to the Weyl tensor.

In this paper, we consider a class of black hole in $F(R)$ gravity with trace anomaly and look for analytical solutions. In order to study the basics solutions of general $F(R)$ gravity theories with constant curvature solutions $R = R_0$ with

*Electronic address: hendi@mail.yu.ac.ir

†Electronic address: d.momeni@yahoo.com

CA, let us start from the action

$$S = S_g + S_m, \quad (1)$$

in which S_g is the 4-dimensional action

$$S_g = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R + f(R)], \quad (2)$$

where $8\pi G = M_p^{-2}$, M_p corresponds with the Planck mass, g is determinant of the metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$), R is the scalar curvature and $R + f(R)$ is the function defining the theory under consideration. As the simplest example, the Einstein gravity with cosmological constant Λ is given by $f(R) = -2\Lambda$. We know that one loop quantum corrections lead to a trace anomaly of the energy-momentum (EM) tensor of conformal field theories. In general, as we can show that, the trace anomaly has the form [14]

$$\langle T_{\mu\nu} \rangle = g_{\mu\nu} [\beta C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} - \alpha(R^2 - 4R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})], \quad (3)$$

where α, β are two positive constants depending on the degrees of freedom of quantum fields and their values are not important for our discussions. The first term is a polynomial of Weyl tensor (B anomaly), while the second (Gauss-Bonnet) term is Euler characteristic of the 4-dimensional spacetime (A anomaly). From the above considerations, the equations of motion (EOM) in the metric formalism are just

$$R_{\mu\nu} (1 + f_R) - \frac{1}{2} [R + f(R)] g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\mu \nabla^\mu) f_R - 8\pi G g_{\mu\nu} [\beta C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} - \alpha(R^2 - 4R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})] = 0, \quad (4)$$

where $f_R = \frac{df(R)}{dR}$. The problem of finding the general static spherically symmetric (SSS) solution in arbitrary $f(R)$ theories without imposing the constant curvature condition is in principle too complicated. The required condition to get vacuum constant curvature solutions $R = R_0$ (from now R_0 will denote a constant curvature value) in vacuum implies

$$R_0 (1 + f_{R_0}) = 2 [R_0 + f(R_0)] \quad (5)$$

For this kind of solutions, an effective cosmological constant may be defined as $\Lambda_D^{eff} = \frac{R_0}{4}$. This type of black-hole discussed in [25]. It has been proved that the only static and spherically symmetric vacuum solution (SSVS) with constant curvature of any $F(R)$ gravity is just the Hawking- Page black hole in AdS space [26]. But for the theory with trace anomaly with $T = T_\mu^\mu$ this condition is

$$R(1 + f_R) - 2[R + f(R)] - 8\pi GT = 0 \quad (6)$$

One can consider (6) as a differential equation (DE) for the $f(R)$ function and with a $T = T(r) \equiv T(R)$. The general solution for DE (6) is given by

$$f(R) = -R + c_1 R^2 + 8\pi G R^2 \int^R \frac{T(R')}{R'^3} dR' \quad (7)$$

II. ASSUMPTION OF EM TENSOR WITH TOPOLOGICAL METRIC

We consider the external metric for the gravitational field produced by a non rotating object in $f(R)$ -Weyl (CA) gravity. The general 4-dimensional topological metric can be written as [27]

$$ds^2 = \lambda(r) dt^2 - \mu^{-1}(r) dr^2 - r^2 (d\theta_1^2 + \Omega^2 d\theta_1^2). \quad (8)$$

where Ω denotes $\sin \theta_1$, 1 and $\sinh \theta_1$ for $k = 1$ (spherical horizon), $k = 0$ (flat horizon), and $k = -1$ (hyperbolic horizon), respectively. We follow the methodology of Cai et.al [21]. We making an assumption that to have a non trivial black hole solution with metric (8) at a black hole horizon, say r_H , the EM tensor must satisfy the auxiliary condition, $\langle T_t^t(r) \rangle_{r=r_H} = \langle T_r^r(r) \rangle_{r=r_H}$, which holds not on horizon even in the whole spacetime. We take the EM tensor for SSS configurations as

$$T_\mu^\nu = \text{diag}(\rho(r), -p(r), -p_\perp(r), -p_\perp(r)), \quad (9)$$

where its trace is

$$T = \rho(r) - p(r) - 2p_{\perp}. \quad (10)$$

Applying the auxiliary condition to metric (8), one can easily show that $\lambda(r) = \mu(r)$. Our main goal is searching for an exact solution for (4) with definite $F(R)$ action given by (7) and determining the metric functions of metric (8). The trace T from Eq. (10) is not constant since the components of the EM tensor (9) are functions of the radial coordinate r . Thus in general, the $f(R)$ functions are very different depending on the form of the T from (10). Locally, let us to take T as a constant which we can find that Eq. (7) reduces to

$$f(R) = -R + c_1 R^2 - 4\pi G T. \quad (11)$$

This action is a second order corrected action of the Einstein-Hilbert action with a cosmological constant $\Lambda = 2\pi G T$. Thus it's solution is a second order corrected (anti) de Sitter spacetime for a typical (negative) positive value of T . Since the Lagrangian of Einstein gravity (R -term) remove in this model (11), the corresponding solutions are not important for us. In next section we will convert the Eq. (4) to a more convent form.

III. NEAR HORIZON AND APPROXIMATE SOLUTIONS FOR CONSTANT CURVATURE

In this section we focuss on the special case $f(R) = -2\Lambda$, with constant curvature i.e. $R = R_0$. Firstly it is adequate to rewrite the EM (3) as the following form

$$< T_{\mu\nu} > = g_{\mu\nu} \left[\left(\frac{\beta}{3} - \alpha \right) R^2 - 2(\beta - 2\alpha) R_{\alpha\beta} R^{\alpha\beta} + (\beta - \alpha) R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right], \quad (12)$$

where we used the following identity [28]

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = \frac{1}{3} R^2 - 2R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}. \quad (13)$$

As we pointed former, α, β are two constants depending on the degrees of freedom of quantum fields and their values are not important for our discussion. We choice $\alpha = \beta$ and also define a new parameter $\alpha' = 2\alpha$. Thus, we write the next EOM

$$\begin{aligned} R_{\mu\nu}(1 + f_R) - \left(\frac{1}{2} [R + f(R)] - \frac{8\pi G \alpha'}{3} R^2 \right) g_{\mu\nu} \\ + (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \nabla_{\beta} \nabla^{\beta}) f_R = 8\pi G \alpha' g_{\mu\nu} (R_{\alpha\beta} R^{\alpha\beta}). \end{aligned} \quad (14)$$

Constant curvature solutions for (14) with $f(R) = -2\Lambda$, can be written as

$$R_{\mu\nu} = \alpha'' g_{\mu\nu} (\gamma + R_{\alpha\beta} R^{\alpha\beta}), \quad (15)$$

where $\gamma = \frac{\frac{1}{2}R_0 - \frac{1}{3}\alpha'' R_0^2 - \Lambda}{\alpha''}$ and $\alpha'' = 8\pi G \alpha'$. Applying the metric (8) with $\lambda = \mu$ to Eq. (15), one leads to the next nonlinear differential equations (NDEs)

$$\lambda''(r) = \frac{1}{r^2} \left[-2r\lambda'(r) \pm \sqrt{2r^4 \left[-\gamma + \frac{1}{8\alpha''^2} \right] - 4[r\lambda'(r) + \lambda(r) - k]^2} \right] - \frac{1}{2\alpha''} \quad (16)$$

$$\lambda''(r) = \frac{1}{r^2} \left[-2r\lambda'(r) \pm \sqrt{2r^4 \left[-\gamma + \frac{-r\lambda'(r) + k - \lambda(r)}{r^2\alpha''} \right] - 4[r\lambda'(r) + \lambda(r) - k]^2} \right] \quad (17)$$

where prime denotes the derivative with respect to the radial coordinate r . We must solve Eqs. (16) and (17) simultaneously. This NDEs have no (simple) exact solution but we can solve it approximately or numerically with a suitable boundary conditions. According to the Hawking-Bekenstein formula for temperature of the black hole, if the metric (8) posses a black hole solution with a horizon located at $r = h$, we can deduce $\lambda(h) = 0$ and

$$T = \frac{\lambda'(h)}{4\pi}. \quad (18)$$

Examining (16) and (17) at horizon we can write

$$\lambda''(h) = \frac{1}{h^2} \left[-8\pi hT \pm \sqrt{2h^4 \left[-\gamma + \frac{1}{8\alpha''^2} \right] - 4(4\pi hT - k)^2} \right] - \frac{1}{2\alpha''} \quad (19)$$

$$\lambda''(h) = \frac{1}{h^2} \left[-8\pi hT \pm \sqrt{2h^4 \left[-\gamma + \frac{-4\pi hT + k}{\alpha'' h^2} \right] - 4(4\pi hT - k)^2} \right] \quad (20)$$

Here, we desire to write near horizon solution of $\lambda(r)$. To do this, we equate Eq. (16) with Eq. (17) and obtain

$$\alpha'' = \alpha_c = -\frac{3h^2 [8\pi hT - 2\Lambda h^2 + R_0 h^2 - 2k]}{2 [12 (4\pi hT - k)^2 - R_0^2 h^4]}. \quad (21)$$

In other word, Eq. (16) is equal to Eq. (17) for specific value of α'' , (21). Thus, the near horizon solution of $\lambda(r)$ of Eqs. (16) and (17) with Eq. (21) is obtained as

$$\lambda(r) \simeq 4\pi T(r - h) + \frac{(r - h)^2}{2h^2} \left[-8\pi hT \pm \sqrt{2h^4 \left[-\gamma + \frac{-4\pi hT + k}{\alpha_c h^2} \right] - 4(4\pi hT - k)^2} \right]. \quad (22)$$

On the other side, for small values of α'' , we can expand Eqs. (16) and (17) and solve the resulting DEs with the following solution

$$\lambda(r) = k - \frac{2M}{r} + \frac{r^2}{6} (2\Lambda - R_0). \quad (23)$$

Identifying this metric function with a Schwarzschild-(a)ds metric gives

$$R_0 = 4\Lambda, \quad (24)$$

and thus, for small values of α'' , the metric (8) reads as

$$ds^2 = \left[k - \frac{2M}{r} + \frac{r^2}{6} (2\Lambda - R_0) \right] dt^2 - \frac{dr^2}{\left[k - \frac{2M}{r} + \frac{r^2}{6} (2\Lambda - R_0) \right]} - r^2 d\Omega^2 \quad (25)$$

IV. EXACT SOLUTIONS

A. $f(R) = -2\Lambda$ model:

1. Case I: $\alpha \neq 0$, $\beta = 0$

At the first step and following the procedure in [21], we consider $f(R) = -2\Lambda$ with type A anomaly (Gauss-Bonnet term). It is easy to show that for $\lambda(r) = \mu(r)$, the field equation (4) reduces to

$$\frac{d}{dr} [\Pi + \Pi_{ii}] = 0, \quad (26)$$

where

$$\Pi = 2\alpha'' \lambda'(r) [\lambda(r) - k] + \frac{\Lambda r^3}{3}, \quad (27)$$

$$\Pi_{ii} = \begin{cases} r [\lambda(r) - k] & i = t, r \\ \frac{r^2}{2} \lambda'(r) & i = \theta, \phi \end{cases}, \quad (28)$$

Using Eq. (26) with Eqs. (27) and (28), the metric function can be obtained as

$$\lambda(r) = k - Ar^2, \quad (29)$$

where

$$A = \frac{3 \pm \sqrt{9 - 48\Lambda\alpha''}}{24\alpha''}. \quad (30)$$

We should note that for $\alpha'' = 3/(16\Lambda)$, Eq. (30) reduces to $2\Lambda/3$ and the presented solution is close to asymptotic (a)dS solution.

2. Case III: arbitrary $\alpha = \beta$

In what follows, we obtain an exact solution of the field equation (16) with A and B anomalies, simultaneously with $f(R) = -2\Lambda$, which is Einstein- Λ gravity with CA. Straightforward calculations show that, in this model, one can obtain topological Schwarzschild solution as [23]

$$\lambda(r) = \mu(r) = k - Ar^2 - \frac{2M}{r}, \quad (31)$$

where A is given in Eq. (30) and $\beta'' = 16\pi G\beta = \alpha''$. As we mentioned before, one can obtain approximate Schwarzschild-(a)dS solution, provided the parameters of the solution are chosen suitably.

B. $f(R) = c_1 R^2 - 2\Lambda$ model:

Considering a linear function of $T(R') = cR' + b$, we find that Eq. (7) leads to the Einstein-Hilbert action with a R^2 correction with a cosmological constant $\Lambda = 2b\pi G$ and $c = \frac{-1}{8\pi G}$. Straightforward calculations show that Eq. (31) is a solution of field equation (4) for $\alpha'' = \beta''$ and obtained A (30).

We note that asymptotic flat Schwarzschild solution may be found by setting $k = 1$ and $b = 0$, which exactly yields the vanishing cosmological constant. In other word, R^2 correction does not effect on the asymptotic behavior of the solutions.

In addition to the Schwarzschild solution, one can show that topological charged solution is another exact solution of this model. In other word, we should note that

$$\lambda(r) = \mu(r) = k - Ar^2 + \frac{q^2}{r^2}, \quad (32)$$

is a solution of FE (4), provided the parameters of the solution are chosen as follow

$$\beta'' = \frac{5\alpha''}{6}, \quad (33)$$

$$c_1 = \frac{-1}{24A}, \quad (34)$$

where A is the same as Eq. (30).

It is notable that the presented uncharged black holes is the same as solutions which obtained in Ref. [30] for $\alpha = 0$. Although for $g_{tt} = g_{rr}^{-1}$ in Ref. [30], the metric function has a charged term $\frac{C_1}{r^2}$ in analogy with our charged solution, but these solutions are completely different. They have different geometry and Ricci scalar R with various asymptotic behavior.

Here, we want to discuss about the stability of such solution as an example of $F(R)$ gravity solutions with CA. The stability of the solutions in $F(R)$ gravity has been discussed in literatures [31]. In fact, the stability of the de Sitter solution may be obtained by imposing the one-loop effective action to be real. In order to study of stability, some various techniques have been employed which all these methods are in agreement with the following conditions which state the existence and the stability of the de Sitter solution:

$$\begin{aligned} I &: 2F(R_0) = R_0 F'(R_0), \\ II &: \frac{F(R_0)}{F'(R_0)} > 0, \\ III &: \frac{F'(R_0)}{R_0 F''(R_0)} > 1. \end{aligned} \quad (35)$$

Equations *I* and *II*, state the existence of a solution with positive constant curvature, while equation *III* ensures the stability of such a solution. We should note that one can obtain such conditions by a classical perturbation method [32].

Here we use these conditions to our de Sitter solutions. In this model one has

$$\begin{aligned} F(R) &= R - 2\Lambda + c_1 R^2, \\ F'(R) &= 1 + 2c_1 R, \\ F''(R) &= 2c_1. \end{aligned}$$

Substitute de Sitter solution in which $R_0 = 4\Lambda$ ($\Lambda > 0$), we obtain

$$\begin{aligned} F(R_0) &= 2\Lambda + 16c_1\Lambda^2, \\ F'(R_0) &= 1 + 8c_1\Lambda, \\ F''(R_0) &= 2c_1. \end{aligned}$$

which confirm that the first and second conditions of Eq. (35) satisfied and for third condition we get

$$1 + \frac{1}{8c_1\Lambda} > 1,$$

As a result, de Sitter solution of this model is stable for $c_1 > 0$.

C. $f(R) = c_1 R^2 + K \ln(R) - 2\Lambda$ model:

The cosmological evolution of the $f(R)$ models based on logarithmic correction has been studied and it has been claimed that such theories have a well-defined Newtonian limit [33]. Here, we desire to consider R^2 with logarithmic corrections, simultaneously. To do this, one can consider $T(R') = cR' + a_1 \ln(R') + a_2$ in Eq. (7), which leads to $f(R) = c_1 R^2 + K \ln(R) - 2\Lambda$, when

$$c = -\frac{1}{8\pi G} \quad (36)$$

$$a_1 = -\frac{K}{4\pi G}, \quad (37)$$

$$a_2 = \frac{K + 4\Lambda}{8\pi G}. \quad (38)$$

Considering the mentioned $f(R)$, one can show that Eq. (31) is a solution of field equation for

$$\beta'' = \alpha'', \quad (39)$$

$$K = \frac{4(\Lambda + 12A^2\alpha'' - 3A)}{2\ln(12A) - 1}. \quad (40)$$

It is notable that for $k = 1$, one may found asymptotic dS solution for specific value of conformal parameter $\alpha'' = \frac{3K}{16\Lambda^2} [2\ln(4\Lambda) - 1]$.

Now, we search about charged solution. Considering the mentioned model with Eq. (40), it is easy to show that Eq. (32) with $\beta'' = 5\alpha''/6$ satisfy the FE (4), provided

$$c_1 = \frac{-1}{72} \frac{\Lambda + 12\alpha''A^2 + 6A[\ln(12A) - 1]}{A^2(2\ln(12A) - 1)}. \quad (41)$$

D. $f(R) = c_1 R^2 + c_2 R^n - 2\Lambda$ model with $n > 2$:

Although most of $f(R)$ gravity models are well-behavior in the weak gravity regime [34], But for the strong gravity regime, some of these models have a serious drawback such as singularity problem. In Ref. [35], Kobayashi and Maeda has been resolved the singularity problem arising in the strong gravity by considering a higher curvature correction term proportional to R^n . In order to obtain the mentioned $f(R)$ model (R^2 and R^n corrections to Einstein-Hilbert action), one may consider $T(R') = \frac{-1}{8\pi G} R' + \frac{(n-2)c_2}{8\pi G} R'^n + \frac{\Lambda}{2\pi G}$ in Eq. (7).

To obtain Schwarzschild solution Eq. (31) in this model, it is sufficient to consider $\beta'' = \alpha''$ with

$$c_2 = \frac{-4(\Lambda + 12A^2\alpha'' - 3A)}{(n-2)(12A)^n}, \quad (42)$$

which confirm that depend on the sign of Λ , one may achieve asymptotic adS/dS solutions provided $\alpha'' = -3c_2(n-2)(4\Lambda)^{n-2}$.

Considering $\beta'' = 5\alpha''/6$ with Eq.(42), it is straightforward to obtain charged solution Eq. (32) for

$$c_1 = \frac{1}{72} \frac{n\Lambda + 12n\alpha''A^2 - 6A(n-1)}{A^2(n-2)}. \quad (43)$$

We should note that the former solution is asymptotically dS (adS) for $\Lambda > 0$ ($\Lambda < 0$) if we set the free parameters of this model in the following manner

$$c_1 = -\frac{1}{8\Lambda} + \frac{n\alpha''}{6(n-2)}, \quad (44)$$

$$c_2 = -\frac{\alpha''}{3(n-2)(4\Lambda)^{n-2}}. \quad (45)$$

V. PROPERTIES OF THE EXACT SOLUTIONS

Now, let us look for the singularity of the solutions. Considering Schwarzschild-(a)dS like (31) with metric (8), one can show that, the Kretschmann scalar is given by

$$R_{\mu\nu\kappa\eta}R^{\mu\nu\kappa\eta} = 24A^2 + \frac{48M^2}{r^6}, \quad (46)$$

therefore there is a singularity located at $r = 0$ (with nonzero $\alpha = \beta$), as the Schwarzschild solution in general relativity.

In addition, it is easy to show that in the case of charged solution (32), the Kretschmann scalar is

$$R_{\mu\nu\kappa\eta}R^{\mu\nu\kappa\eta} = 24A^2 + \frac{56q^4}{r^8}, \quad (47)$$

which confirm that the geometry of singularity ($r \rightarrow 0$) is the same as Reissner-Nordström black hole. It is notable that presented charged black hole solution (32), has two horizons for hyperbolic horizon ($k = -1$) and for $k = 0, 1$, we encounter with naked singularity (see Fig. 1 for more details).

It is notable that the presented solutions are nonsingular for vanishing mass (M) and charge (q).

VI. VIABILITY CONDITIONS OF $F(R)$ THEORIES

The first study on cosmological viable $f(R)$ models have been presented in Ref. [36]. The fundamental conditions and restrictions [37] that are usually imposed to action

$$S_g = \frac{1}{16\pi G} \int d^4x \sqrt{-g} F(R) \quad (48)$$

theories to provide consistent both gravitational and cosmological evolutions are:

1. $F''(R) \geq 0$ for $R \gg F''(R)$. This is the stability requirement for a high curvature classical regime [38] and that of the existence of a matter dominated era in cosmological evolution. A simple physical interpretation can be given to this condition: if an effective gravitational constant $G_{eff} \equiv G/(1 + F'(R))$ is defined, then the sign of its variation with respect to R , dG_{eff}/dR , is uniquely determined by the sign of $F''(R)$, so in case $F''(R) < 0$, G_{eff} would grow as R does, because R generates more and more curvature itself. This mechanism would destabilize the theory, as it wouldn't have a fundamental state because any small curvature would grow to infinite. Instead, if $F''(R) \geq 0$, a counter reaction mechanism operates to compensate this R growth and stabilize the system.

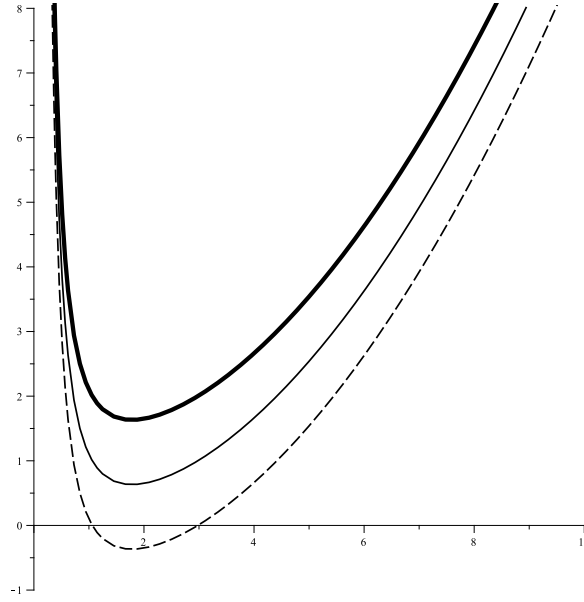


FIG. 1: $\lambda(r)$ (Eq. (32)) versus r for $A = -0.1$, $q = 1$, and $k = 1$ (bold line), $k = 0$ (solid line) and $k = -1$ (dashed line).

2. $1 + F'(R) > 0$. This condition ensures that the effective gravitational constant is positive, as it can be checked from the previous definition of G_{eff} . It can also be seen from a quantum point of view as the condition that avoids the graviton from becoming a ghost [39].
3. $F'(R) < 0$. Keeping in mind the strong restrictions of Big Bang nucleosynthesis and cosmic microwave background, this condition ensures GR behavior to be recovered at early times, that is, $F(R)/R \rightarrow 0$ and $F'(R) \rightarrow 0$ as $R \rightarrow \infty$. Conditions 1 and 2 together demand $F(R)$ to be a monotone increasing function between the values $-1 < F'(R) < 0$.
4. $F'(R)$ must be small in recent epochs. This condition is mandatory in order to satisfy imposed restrictions by local (solar and galactic) gravity tests. As the analysis done in [40] indicates, the value of $|F'(R)|$ must not be bigger than 10^{-6} (although there is still some controversy about this). This is not a needed requirement if the only goal is to obtain a model that explains cosmic acceleration.

Now we examine the viability conditions (1-4) for a general $F(R) = R + f(R)$ model described in (7). Explicitly, we have

$$F'(R) = 2c_1 R + 16\pi G R \int^R \frac{T(R')}{R'^3} dR' + 8\pi G \frac{T(R)}{R} \quad (49)$$

$$F''(R) = 2c_1 + 8\pi G \frac{T(R)}{R^2} + 16\pi G \int^R \frac{T(R')}{R'^3} dR' \quad (50)$$

Validity of the conditions (1-4) depends on the form of the $T(R)$. We will summarize the different models of $F(R)=R+f(R)$ discussed in this paper as the follows. Note that since we considered the $F(R) = R + f(R)$ models, thus in each case we firstly written the explicit form of the $F(R)$.

A. $T(R)=T=c.t.e$

In this case, one can obtain

$$F'(R) = 2c_1 R \quad (51)$$

$$F''(R) = 2c_1 \quad (52)$$

with the following properties:

1. $F''(R) = 2c_1 R \geq 0$ for $R \gg 2c_1$. Thus this model is stable or a high curvature classical regime. The effective gravitational constant is $G_{eff} \equiv \frac{G}{1+2c_1 R}$. In the early universe epoch, when $R \gg R_0 = 2c_1$, this effective G reduces to $G_{eff} \simeq \frac{1}{R}$, its variation with respect to R , $\frac{dG_{eff}}{dR} < 0$. Thus G_{eff} would decrease as R grows in the early times, this mechanism would stabilize the theory, as it would have a fundamental state.
2. Obviously for $R \gg 2c_1, 1 + F'(R) > 0$. It means that the G_{eff} is positive. Thus this case of theory is free from ghosts.
3. $F(R)/R \rightarrow \infty$ and $F'(R) \rightarrow \infty$ as $R \rightarrow \infty$. Thus only for $c_1 = 0$, this model recovers GR behavior at early times.
4. In early times, $F'(R) \ll 1$. Thus our model passes the local solar system tests. Further, from $|2c_1 R| < 10^{-6}$ gives the bound on $|c_1| < \frac{10^{-6}}{R}$, $R \approx O(\Lambda)$.

$$\mathbf{B.} \quad f(R) = -2\Lambda$$

This case describes the Einstein-Hilbert action with a cosmological constant with

$$F'(R) = 1 \tag{53}$$

$$F''(R) = 0 \tag{54}$$

1. $F''(R) = 1 \geq 0$ for $R \in \mathbb{R}$. Thus this model is stable.
2. Obviously $1 + F'(R) > 0$. It means that the $G_{eff} = G$ is positive. Thus this case of theory is free from ghosts. Further, it coincides with GR at the action level.

$$\mathbf{C.} \quad f(R) = c_1 R^2 - 2\Lambda$$

Now, we have $F(R) = R + c_1 R^2 - 2\Lambda$ and thus we've

$$F'(R) = 1 + 2c_1 R \tag{55}$$

$$F''(R) = 2c_1 \tag{56}$$

1. $F''(R) = 2c_1 \geq 0$ for $c_1 > 0$. This is the stability requirement for a high curvature classical regime of this model and that of the existence of a matter dominated era in cosmological evolution. This model induces an effective gravitational constant $G_{eff} \equiv G/2(1 + c_1 R)$ is defined, then the sign of its variation with respect to R , $\frac{dG_{eff}}{dR} = -\frac{Gc_1}{2(1+c_1 R)^2}$, is uniquely determined by the sign of c_1 , so in case $c_1 < 0$, G_{eff} would grow as R does, because R generates more and more curvature itself. The problem changed when the sign of the c_1 reverses. This mechanism would destabilize the theory, as it wouldn't have a fundamental state because any small curvature would grow to infinite. Instead, if $c_1 \geq 0$, a counter reaction mechanism operates to compensate this R growth and stabilize the system.
2. $1 + F'(R) = 2(1 + c_1 R) > 0$. This condition ensures that the effective gravitational constant is positive and from a quantum point of view as the condition for avoiding from a ghost graviton.
3. $2c_1 R < 0$ for $c_1 < 0$. This condition ensures GR behavior to be recovered at early times.
4. $F'(R) = 1 + 2c_1 R$ must be small in recent epochs. This condition is mandatory in order to satisfy imposed restrictions by local (solar and galactic) gravity tests. The value of $|1 + 2c_1 R|$ must not be bigger than 10^{-6} .

$$\mathbf{D.} \quad f(R) = c_1 R^2 + K \ln(R) - 2\Lambda$$

In this case the Lagrangian of the model is $F(R) = R + c_1 R^2 + K \ln(R) - 2\Lambda$, we have

$$F'(R) = 1 + 2c_1 R + \frac{K}{R} \tag{57}$$

$$F''(R) = 2c_1 - \frac{K}{R^2} \tag{58}$$

1. $F''(R) = 2c_1 - \frac{K}{R^2} \geq 0$ for $R \geq \sqrt{\frac{K}{2c_1}}$. This is the stability requirement for a high curvature classical regime of this model and that of the existence of a matter dominated era in cosmological evolution. This model induces an effective gravitational constant $G_{eff} \equiv G/(2 + 2c_1R + \frac{K}{R})$ is defined, then the sign of its variation with respect to R , $\frac{dG_{eff}}{dR} = -\frac{G(2c_1 - \frac{K}{R^2})}{(2 + 2c_1R + \frac{K}{R})^2}$, is uniquely determined by the sign of c_1, K , so in case $c_1 < 0, K > 0$, G_{eff} would grow as R does, because R generates more and more curvature itself. The problem changed when the sign of the c_1, K reverses. This mechanism would destabilize the theory, as it wouldn't have a fundamental state because any small curvature would grow to infinite.
2. $1 + F'(R) = 2 + 2c_1R + \frac{K}{R} > 0$. This conditions ensures that the effective gravitational constant is positive and from a quantum point of view as the condition for avoiding from a ghost graviton. The root of the equation $1 + F'(R) = 2 + 2c_1R + \frac{K}{R} = 0$ locates at $R_{\pm} = \frac{-1 \pm \sqrt{1 - 8c_1K}}{4c_1}$. This is the critical curvature. If $c_1 > 0$ then the model is free from ghosts for $R > R_+, R < R_-$. But if $c_1 < 0$ then the ghosts are present for this range of the curvature.
3. $1 + 2c_1R + \frac{K}{R} < 0$ for $c_1 < 0, R > R_+, R < R_-$. This condition ensures GR behavior to be recovered at early times.
4. $F'(R) = 1 + 2c_1R + \frac{K}{R}$ must be small in recent epochs only for small negative values of c_1 . This condition is mandatory in order to satisfy imposed restrictions by local (solar and galactic) gravity tests. The value of $|1 + 2c_1R + \frac{K}{R}|$ must not be bigger than 10^{-6} .

E. Model with $f(R) = c_1R^2 + c_2R^n - 2\Lambda$ with $n > 2$

In this case, we have $F(R) = R + c_1R^2 + c_2R^n - 2\Lambda$. Explicitly, we have

$$F'(R) = 1 + 2c_1R + nc_2R^{n-1} \quad (59)$$

$$F''(R) = 2c_1 + n(n-1)c_2R^{n-2} \quad (60)$$

Validity of the conditions (1-4) depends on the signs of the constants c_1, c_2 for $n > 2$. If we take $c_1 > 0, c_2 > 0$, then obviously for all values of the $R, F'(R) \geq 0, F''(R) \geq 0$. The conditions (1,2) are satisfied automatically. But for $c_1 < 0$ it is possible to have $F'(R) < 0$. Thus the model satisfy the local solar system tests for some values of the c_1, c_2 . For example take $n=3$. Then the condition $F'(R) < 0$ can be satisfied by such values of the curvature in the interval $R_- < R < R_+$ for $c_2 > 0, c_1^2 > 3c_2$ and $R > R_+$ or $R < R_-$ for $c_2 < 0$ where $R_{\pm} = -c_1 \pm \sqrt{c_1^2 - 3c_2}$.

F. On the validity of CA coefficients

Anomaly is a classical symmetry breaking by quantum corrections. It seems that by such a definition, this effect has significance role only in early universe. But now, we show that in a universe which is in accelerating expansion phase, filled by dark energy, the phenomena can be described by CA terms. The main idea belongs to [41]. We summarized here the main results (for details refer to the [41] and references there). As we know that, the CA term, if it is considered as an IR contribution to the stress-energy, helps to understanding of behavior of the EM tensor of the matter fields at low energy of any effective theory. As we know that, the trace term $\langle T_{\mu}^{\mu} \rangle$ is of order of the $\langle T_{\mu}^{\mu} \rangle \sim H^4$ where H is the usual Hubble constant. Cut-off from QCD tells us that this value must be of order $\langle T_{\mu}^{\mu} \rangle \sim 10^{-12}(eV)^4$ [42]. This is the observational vacuum energy density. Difference between these values is the first cosmological constant problem and has no satisfactory solution till now. By a systematic analysis of the CA terms corrections to the vacuum energy density it has been shown that under certain conditions the CA terms can provide an appropriate scale observed in nature as predicted by QCD estimates. The first step is writing the total action for the low energy theory which contains the usual Einstein-Hilbert action, the conformally invariant Weyl portion, the higher order curvature invariants and finally the anomalous contribution. By writing the total EM tensor we observe that the resulting expression is correctly proportional to the equations of motion for some auxiliary fields. It is easy to show that the anomalous contribution to the stress tensor and the Weyl one would be of the same order of magnitude. If the spacetime been conformally flat, then the Weyl invariant term vanishes, we have only the anomalous contribution as the only unique contribution to the energy tensor. Consider the spacetime as the static de Sitter. By calculating the finite contribution to the energy and pressure density, it is shown that the result does not depend on the CA coefficients. Thus we remove the divergences of the CA stress tensor. Indeed, the contribution of the CA

terms reduces to the local, second order in the curvature, geometrical terms, which are of order H^4 , and therefore irrelevant at scales lower than the Planck one.

VII. CONCLUSION

It has been shown that $F(R) = R + f(R)$ gravity has (a)dS Schwarzschild solution. In addition, charged solutions of some pure $F(R)$ models have been investigated in Ref. [5]. In the present paper, we considered CA as a source of $F(R)$ gravity and investigated its consequences.

At first, we used temperature conception of black holes to find a near horizon solution and then we found an approximate solution which one might interpret it as a Schwarzschild-(a)dS solution.

Next step is devoted to find exact solutions. Calculations showed that depending on the values of free parameters, one may find Schwarzschild and charged solutions. Also, we found that the geometry of uncharged (charged) singularity ($r \rightarrow 0$) is corresponding to the Schwarzschild (Reissner-Nordström) black hole.

Finally, we investigated the fundamental conditions to obtain viable $F(R)$ gravity. In other word, we found some restrictions on free parameters to guarantee stability as well as ghosts free conditions. In addition, we checked the behavior of the $F(R)$ theory at early times and also satisfaction of local (solar and galactic) gravity tests.

It is worthwhile to investigate the thermodynamics properties and also generalize our solutions to higher dimensional spacetime, and these problems are left for the future.

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